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LARGE AMPLITUDE RESPONSE OF COMPLEX STRUCTURES DUE TO HIGH INTENSITY NOISE

Structural Integrity Branch
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NOTICE

This technical report has been reviewed and is approved for publication.

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A problem of interest to the Air Force is the design of acoustically sound aircraft structural components. This is because sonic fatigue failures have resulted in unacceptable maintenance and inspection burdens associated with the operation of the aircraft. In some instances, sonic fatigue failures have resulted in major redesign efforts of structural components. Currently, the sonic fatigue design methods are based upon a combination of experimental and analytical techniques. The analytical methods are based on the linear or

20. Abstract (Continued)

small deflection theory (Sonic Fatigue Design Guide for Military Aircraft, AFFDL-TR-74-112, for example). But, on the contrary, the test structural panels respond nonlinearly with large deflections at high intensity acoustic pressure levels. This large amplitude geometrical nonlinearity is the major factor that causes disagreement between the computed and the measured random responses. To improve the analytical design methods, large deflection or nonlinear structure theory must be employed in the analysis. This report presents a review of existing analytical and numerical methods on random excitation on nonlinear multi-degree-of-freedom systems, and an evaluation of these methods based on some realistic considerations from the point of view of their application to complex panel configurations of aircraft structure. These are the Fokker-Planck equation method, equivalent linearization technique, perturbation approach, finite difference method, finite element-equivalent linearization approach, etc. Then, a mathematical formulation, which is based on finite element method and equivalent linearization technique, for complex structural panels subjected to high noise environment is developed. Statistical responses of nodal deflections and element strains using a single-mode approximation are given in terms of the linear frequency, spectral density of excitation pressure, equivalent linear frequency, a strain-deflection transformation matrices. Extension of the quasi-linearization method, which has been used successfully in analysis of large amplitude vibrations of complex structures, to the present nonlinear random excitation problem is also discussed briefly.

FOREWORD

This report summarizes the results of the studies made on the nonlinear response of complex structural panels subjected to broadband random acoustic excitation by the author during his ten week stay at AFFDL/FBED, W-PAFB, OH 45433. The work was supported by the Air Force Office of Scientific Research through the USAF-ASEE Summer Faculty Research Program (W-PAFB), Contract F44620-76-C-0052, The Ohio State University, Columbus, OH 43210.

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NOMENCLATURE

$[c]$	Element damping matrix
$[C]$, $[C]_s$	Damping matrices
$\lceil C \rfloor$	Diagonal generalized damping matrix
e	Error of linearization or equation deficiency
$E[\ ]$	Operator denotes the mathematical expectation
$\{f\}$	Element nodal force vector
$\{F\}$, $\{F\}_s$	Nodal force vectors
g	Structural damping coefficient
$H(\Omega)$	Frequency response function
$[k]$	Element stiffness matrix
$[K]$, $[K]_s$	Stiffness matrices
$\lceil K \rfloor$	Diagonal generalized stiffness matrices
$[k^g]$	Element geometrical stiffness matrix
$[K^g]$, $[K^g]_s$	Geometrical stiffness matrices
K^{eq}	Equivalent linear stiffness constant
$[m]$	Element consistent mass matrix
$[M]$, $[M]_s$	Consistent mass matrices
$\lceil M \rfloor$	Diagonal generalized mass matrix
$\{P\}$	Force vector in modal coordinates
$\{q\}$	Vector of node displacements normal to the surface of structure
$\{q\}_s$	Nodal displacement vector
$[\overline{q_j q_k}]$	Deflection covariance matrix

NOMENCLATURE (CONCLUDED)

$[S]_1$, $[S]_2$	Strain-deflection transformation matrices
$[S_F(\Omega)]$	Cross spectral density matrix of $\{F\}$
$S_p\ (\Omega)$	Spectral density function of P
$\{\delta\}$	Element nodal displacement vector
$\{\varepsilon\}$	Element strain vector
$[\overline{\varepsilon_r \varepsilon_s}]$	Strain covariance matrix
ζ	Damping ratio
λ,μ	Proportionality constant between damping and stiffness and inertia, respectively
$\{\xi\}$	Amplitude vector in modal coordinates
$\{\phi^{(j)}\}$	j-th normalized eigenvector
$[\phi]$	Modal transformation matrix
ω	Linear undamped frequency
ω_{eq}	Equivalent linear frequency
Ω	Angular frequency

SECTION I

INTRODUCTION

The response of outside surfaces or skins of an aircraft structure to high intensity acoustic pressure levels has been the subject of considerable research effort (References 1 and 2, for example). The complex problem may be separated into three parts:

(1) Prediction of the acoustic loading,

(2) Determination of the response of complex structural panels to this excitation, and

(3) Estimation of the fatigue life.

There are considerable data available to predict the acoustic loads on an aircraft structure due to the many possible sources of high sound levels. These sources are normally classified as to propulsion system noise sources and aerodynamic noise sources. Noise prediction methods for various sources have been summarized in two excellent reports (References 2 and 3).

Basic design considerations and procedures to estimate the fatigue life, various cumulative damage theories, and fatigue curves describing the S-N characteristics of various materials are given in References 1 and 2. Fatigue design data for bonded aluminum structures can be found in Reference 4.

The present study concentrates on the response aspects of the problem, where considerable overlapping also occurs. Special effort is made to incorporate the nonlinear effects due to large deflections into the analysis. The discrepancy between the measured and computed responses can be attributed to these nonlinear effects. The computed results have been based on linear or small deflection theory, while test panels responded nonlinearly with large deflection at sufficiently high intensity noise levels. These are summarized briefly in Section II.

In approaching this problem of random excitation of nonlinear systems, one can turn to a number of prior investigations. Therefore, an important task of this study is to review, classify, and evaluate these existing techniques. The evaluation is based on some realistic considerations from the point of view of their application to complex panels of aircraft structure. These are presented in Section III. It is evident from the survey that additional research is necessary to accurately predict the nonlinear random response of aircraft panels.

Section IV gives a mathematical formulation, which is based on the finite element method and the equivalent linearization approach, for complex structures subjected to random acoustic excitation. An iterative scheme used in the solution procedure and a simplified flow chart are also presented. The ratio of root mean-square (RMS) response of nonlinear and linear cases is expressed as a function of linear frequency, equivalent linear frequency, and power spectral density (PSD) of the excitation. A preliminary procedure for the improvement of predicting responses using large deflection theory is also included in this section.

Section V gives a review of the advances in the development of geometrical stiffness matrices for various finite elements. These matrices are required in the nonlinear random vibration analysis.

A "Quasi-Linearization" method which is, at present, in the conceptual phase is discussed briefly in Section VI. Conclusions and recommended future work are given in Section VII.

SECTION II

EVIDENCE OF LARGE AMPLITUDE NONLINEARITY

Some aspects of current knowledge about the response of structural panels to high intensity noise are discussed in this section. Tests on aircraft structural components have displayed behavior that is not consistent with linear or small deflection theory assumptions. The deviations, which differ for various structural configurations, are suggestive of two types of nonlinearity sources:

(1) Nonlinear damping ratio, and

(2) Nonlinearity due to large deflections.

Commonly used methods for determining damping ratio are the bandwidth method by measuring half-power widths at modal resonances, and the decay rate method by measuring the logarithmic decrement on decaying modal response traces. The values of damping ratio range generally from 0.001 to 0.025 for common panel constructions used in aircraft structure. For such relatively small damping coefficients, it is reasonable to assume that the effects of nonlinear damping on structural behavior would also be small in comparison to large amplitudes. This has been observed in many experiments and is discussed and summarized in the following.

Many documents, for example references 5 to 8, have repeatedly reported that a poor comparison exists between the measured and calculated RMS responses. They all observed that the test panels responded with large deflections at high sound pressure levels, whereas the computed responses were based on linear small deflection theory. This is the major reason for the discrepancy between measured and calculated results. This experimental evidence was taken from:

(1) ASD-TDR-62-26 (Reference 5)

Fitch et al. observed the nonlinear stress response at relatively low siren excitation level for conventional skin-stringer panels as shown in Figure 1. Their explanation to this nonlinear behavior was

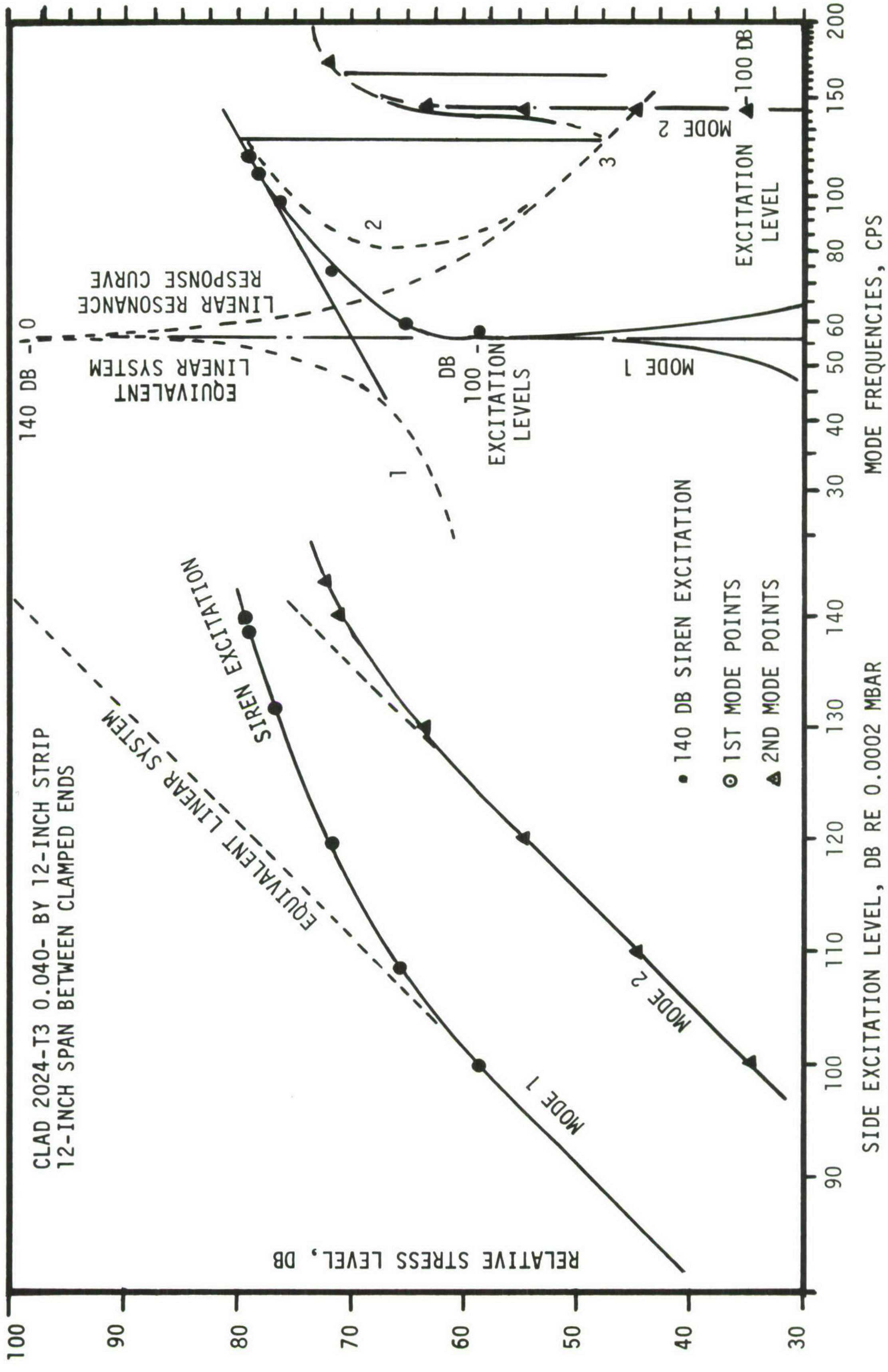


Figure 1. Nonlinear Stress Response (After Fitch et al., Reference 5)

the diaphragm action which limits the amplitude of deflection of the vibrating plates. Examination of Figure 1 indicates that as the excitation level is raised above 104 dB, the structural response begins deviating from the linear assumption.

(2) AFFDL-TR-68-44 (Reference 6)

Three types of typical aerospace structures were tested at an average sound pressure level (SPL) of 157 dB overall. They are a skin-stringer 3-bay, a honeycomb sandwich, and a corrugated sandwich panel. Measured RMS deflection on the skin-stringer panel was surprisingly large, 0.064 inch, or twice the skin thickness. Measured honeycomb panel RMS deflection was 0.031 inch, or 1.4 thickness of the top skin. Comparisons of RMS deflection showed that calculated skin-stringer panel deflection was 34-percent high and calculated honeycomb panel deflection was 5-percent high only. Comparisons between calculated and measured RMS stresses were poor. They are shown in Table 1 taken from Reference 6.

TABLE 1

STRESS COMPARISON
(After Jacobs and Lagerquist, Reference 6)

Panel Type	Stress Component	RMS Stress (kpsi)						
		Calculated	Measured on Panel:					Measured Average
			A	B	C	D	E	
Skin-stringer	$\sqrt{\overline{\sigma_x^2}}$	7.7	2.2	2.9	2.5	---	2.2	2.5
	$\sqrt{\overline{\sigma_y^2}}$	2.4	0.63	0.94	0.78	1.1	0.84	0.87
Honeycomb	$\sqrt{\overline{\sigma_x^2}}$	2.6	2.0	1.8	1.2	1.2	1.3	1.5
	$\sqrt{\overline{\sigma_y^2}}$	2.1	1.3	1.3	0.91	0.84	1.3	1.1

Strain components were also measured with rosette strain gauges mounted on both the upper and lower surfaces. Appreciable membrane stress was recorded, which implies the panels responded with large deflections.

The stress levels measured were actually very low, showing that the nonlinearity is not associated with yield or plasticity of the material, but rather with coupling of inplane and out-of-plane displacements.

(3) AFFDL-TR-71-126 (Reference 7)

Three nine-bay, cross-stiffened, graphite-epoxy panels were exposed in a broad-band 166 dB SPL overall acoustic environment. Under loud speaker excitation, the lowest natural frequency was at 174 Hz. But under the acoustic pressure in the progressive wave test chamber, the lowest frequency increased from 200 Hz to 290 Hz as the SPL was increased from 139 dB to 166 dB. The increase in natural frequency with increasing pressure level is attributed to the large deflections of the panel response. Again strain comparisons were poor as shown in Table 2 taken from Reference 7. Deflection was not measured.

TABLE 2

COMPARISON OF EXPERIMENTAL RESULTS OF CROSS-STIFFENED PANELS WITH RESULTS USING UNSTIFFENED PLATE THEORY
(After Jacobson, Reference 7)

Approach	Method	Fundamental Frequency (Hz)	Strain at $x = 0,\ y = \frac{b}{2}$ (micro-inch/ inch-rms)	Strain at $x = \frac{a}{2},\ y = 0$ (micro-inch/ inch-rms)
Analytic	Simplified Theory (Beam Functions)	182	180(2)	406(2)
Analytic	Finite Element REDYN)	180	180(2)	348(2)
Experi-mental	Test Panel A-GG-B-2	187(1)	96(4)	160(3)
Experi-mental	Test Panel A-GG-B-3	170(1)	74(4)	164(3)

(1) Obtained during damping factor determination under loudspeaker excitation
(2) Strain response to fully correlated, white noise excitation of 1.2 x 10^{-6} psi ^{2}/Hz
(3) Strain gage No. 2 reading during 139 db run
(4) Strain gage No. 7 reading during 139 db run

(4) AFFDL-TR-77-45 (Reference 8)

A total of ten bonded aluminum panels were tested. Table 3, taken from Reference 8, gives the ratio of the fundamental frequency obtained under the loud-speaker excitation to the frequency at which the peak strain PSD occurred in the sonic fatigue test at 166 dB overall SPL. Again, the agreement between the test strains and the predicted strain was poor as shown in Table 4 (taken from Reference 8), and no measured deflection was reported.

Therefore, a conclusion can be reached that any real structure designed for service in high sonic environment regions would respond with large deflection nonlinearity.

TABLE 3

FREQUENCY AND DAMPING DATA
(After Jacobson, Reference 8)

Panel	Damping Factor(1)	Natural Frequencies First Mode	Second Mode	Third Mode	Fourth Mode	f_{166} (2)	$\frac{f_{166}}{f_1}$
		(Hz)	(Hz)	(Hz)	(Hz)	(Hz)	
A-1-1	0.018	137	189	265	340	218	1.59
A-1-2	0.015	97	147	179	266	204	2.10
A-2-1	0.011	144	209	284	386	212	1.47
A-2-2	0.009	141	203	279	376	211	1.50
A-3-1	0.012	165	245	-	-	205	1.21
A-3-2	0.011	141	210	295	413	207	1.47
A-4-1	0.023	80	114	155	226	140	1.75
A-4-2	0.009	84	124	172	235	144	1.71
A-5-1	0.012	103	155	211	292	150	1.46
A-5-2	0.014	99	153	215	300	143	1.44
Average	0.0134						

(1) Nondimensional viscous damping factor, $\frac{c}{c_c}$

(2) The parameter f_{166} was the frequency of the predominant strain response at 166 dB overall SPL.

TABLE 4

STRAIN PREDICTIONS BASED ON THE ASSUMPTION
OF FULLY CLAMPED EDGES OF A PLATE
(After Jacobson, Reference 8)

Test Panel Being Simulated	Over-All SPL	Pressure PSD	RMS Strains					
			Gage No. 2		Gage No. 4		Gage No. 11	
			Test	Predicted	Test	Predicted	Test	Predicted
	(dB)	(psi^2/Hz)	(μ"/")	(μ"/")	(μ"/")	(μ"/")	(μ"/")	(μ"/")
A-2-1	142	5.0×10^{-6}	32	445	27	111	84	270
A-4-1	145	3.4×10^{-5}	75	766	110	192	178	466

SECTION III

REVIEW OF EXISTING APPROACHES ON NONLINEAR RANDOM VIBRATION

Methods used to model structural systems can be basically divided into analytical methods and numerical methods. Analytical approaches are usually focused on obtaining explicit closed-form quantitative results for simple structural configurations. For complex structures such as found in aircraft design, numerical methods must invariably be employed to accurately model the complex configuration. However, analytical methods can generally provide more insight and lead to a better understanding of the problem. And other approximate and numerical techniques can then be extended from there. In this section, both analytical and numerical approaches for solving random excitation of nonlinear systems are reviewed.

1. ANALYTICAL METHODS

a. Fokker-Planck Equation Approach. The most general extension of the Fokker-Planck equation method to the multiple-degree-of-freedom (multiple-DOF) systems of nonlinear second order equations was developed by Caughey (References 9 and 10). One great advantage of this method over all of the other approaches is that it gives an exact solution. However, this should not be construed that all problems relating to the response of nonlinear systems with random excitation have been solved. In fact, exact solutions of the steady-state probability function have only been found for certain restricted classes of problems provided:

(1) The only energy dissipation in the system arises from damping forces which are proportional to the velocity,

(2) The exciting forces are uncorrelated Gaussian white noise,

(3) The spectral density matrix of the excitation is proportional to the damping matrix of the system, and

(4) The restoring force vector of the system is derivable from a potential.

The solution of the time-independent Fokker-Planck equation under these conditions represents a very significant accomplishment. Problems of simple structures which satisfy these four conditions were solved by Herbert (Reference 11 and 12). Yet many problems of practical interest do not satisfy those conditions necessary for a solution. The required relationship between the excitation and the damping matrix is particularly restrictive. In additiion, the transitional probability density function generally can not be found with the Fokker-Planck approach. Without this transitional probability, it is generally impossible to obtain the correlation function and PSD of the response. Thus, a number of approximate techniques have been developed to treat a broader class of problems than is presently possible with the exact analysis. These are the equivalent linearization technique, perturbation method, and others.

b. Equivalent Linearization Approach. This method was originated by Krylov and Bogoliubov (Reference 13). Caughey (Reference 14) and others have extended the equivalent linearization technique to systems of nonlinear differential equations. In Caughey's formulation, the correlation function matrix of the excitation must be diagonalized by the same transformation that diagonalizes the linear mass, damping, and stiffness matrices. This represents a rather severe limitation and in particular precludes the application of this formulation to dynamic systems which are excited randomly at only several nodal points. Lin (Reference 15) used the equivalent linearization method with a single-mode approach and obtained response for a rectangular panel subjected to randomly-varying loadings. Seide (Reference 16) has employed the formulation by Caughey and obtained solution for a simple beam subjected to uniform pressure excitation uncorrelated in time.

Foster (Reference 17), and Iwan and Yang (Reference 18) have extended the equivalent linearization technique by removing the restriction imposed on the transformation. Foster's formulation is very general. He first replaced the original n-DOF second order system by a 2n-DOF first order system. The determination of the equivalent linear stiffness coefficients is then accomplished by the inversion of a 2n x 2n mean-square

matrix and an iterative procedure. Practical application of this approach to a simple deep-ocean tower frame structure is reported in Reference 19.

c. Perturbation Approach. A perturbation method, based on classical perturbation theory, was developed by Crandall (Reference 20) to obtain approximate solutions to nonlinear systems, containing a small parameter, excited by a weakly stationary random Gaussian process. In principle, the perturbation approach can be extended to systems of coupled nonlinear equations in which the nonlinearities contain a small parameter. Lyon (Reference 21) used this method to study the responses of a nonlinear string. Tung, Penzien, and Horonjeff (Reference 22) used the perturbation procedure to a two DOF system. For complex structures, however, the algebraic operations may become so unwieldy that the method is no longer practical. In addition, there are certain subtle questions about the convergence of the power-series expansion for the nonlinear response still remaining unanswered.

d. Other Approximate Methods. Fox et al. (Reference 23) have developed three new approaches: (1) Direct Evaluation of Spectra, (2) Estimates of Equilibrium Distribution, and (3) Generalized Kinetic Equation. They have applied the direct evaluation of spectra method to a hinged uniform beam with the assumptions that the force spectrum is white and that all DOF are equally forced and have the same damping factor. Extension of these methods to complex structures would certainly require considerable efforts.

2. NUMERICAL METHODS

Numerical methods can be subdivided into two categories: numerical solutions to differential equations or finite difference method, and matrix displacement method based on discrete element idealization or finite element method.

a. Finite Difference Approach. Numerical solutions to differential equations are somewhat restricted so that these techniques can be

practically applied only to simple structural configurations. Belz (Reference 24) used the finite difference approach and obtained statistical response for a simple uniform beam subjected to a single concentrated load at the midspan of the beam.

b. Finite Element Method. Application of the finite element methods to <u>linear</u> structures subjected to random excitations have been presented, for example, by Jacobs and Lagerquist (References 6 and 25), Olson and Lindberg (Reference 26), Jacobson (Reference 7), and Olson (Reference 27). In Reference 26, refinement on the continuity between the stiffeners and the panel itself was introduced. Olson in Reference 27 presented a consistent formulation for the cross spectral density matrix of the excitation. Both should improve the accuracy of predicting random responses for linear structures.

Application of the finite element method to deep-ocean towers has been given by Foster (Reference 19) and to off-shore towers by Penzien et al. (Reference 28). For these problems, the nonlinear effects can be expressed explicitly in terms of the displacements or velocities. But, this is not possible for problems of complex panel responses to high intensity noise. The nonlinear effect due to large deflections or the geometrical stiffness matrix is not known <u>a priori</u>.

Extension of the finite element approach to complex structures under high noise environment with large deflection nonlinear effect is the purpose of this research. A matrix formulation which is based on the finite element method and the equivalent linearization technique is developed and presented in the next section.

SECTION IV

MATHEMATICAL FORMULATION

1. FINITE ELEMENT REPRESENTATION

The structures considered will be restricted to stable structural systems which are highly resonant, that is with little damping. These are typical properties for panel components of aircraft structure.

In the matrix structural analysis, the structure is idealized into a finite number of discrete structural elements connected at node points. The physical properties of the structure are assumed to be lumped into individual elements. The stiffness equations of motion for such an element under the influence of dynamic loading, inertia, damping, elastic, and nonlinear large deflection characteristics are:

$$[m]\{\ddot{\delta}\} + [c]\{\dot{\delta}\} + ([k] + [k^g(\{\delta\})])\{\delta\} = \{f(t)\} \qquad (1)$$

where $\{\delta\}$ and $\{f\}$ are vectors of nodal displacements and applied forces, respectively. The consistent mass $[m]$, damping $[c]$, and linear stiffness $[k]$ matrices have been developed for almost every beam, plate, and shell element available. The element geometrical stiffness matrix or nonlinear stiffness matrix $[k^g\ (\{\delta\})]$, which is displacement dependent and is induced due to large deflections, will be discussed in Section V.

By assembling all the elements, and applying the kinetic boundary conditions, the equations of motion of the structure are:

$$[M]_s\{\ddot{q}\}_s + [C]_s\{\dot{q}\}_s + ([K]_s + [K^g(\{q\}_s)])\{q\}_s = \{F(t)\}_s \qquad (2)$$

in which $\{q\}_s$ and $\{F\}_s$ denote the vectors of nodal displacements and forces of the structure, matrices $[M]_s$, $[C]_s$, $[K]_s$, and $[K^g]_s$ are the mass, damping, stiffness, and geometrical stiffness coefficients of the structure, respectively.

Applying the static condensation (or Guyan reduction) and retaining only those DOF, say m of them, normal to the surface of the structure, Equation 2 may be written as:

$$\underset{m\times m}{[M]}\{\ddot{q}\} + [C]\{\dot{q}\} + ([K] + [K^g(\{q\})])\underset{m\times 1}{\{q\}} = \{F(t)\} \qquad (3)$$

where $\{q\}$ is a vector containing all nodal deflections normal to the panel structure. The matrices [M], [C], and $[K^g]$ denote the reduced mass, damping, stiffness, and geometrical stiffness, respectively.

2. DAMPING REPRESENTATION

For certain forms of damping, the coupled nonlinear equations of motion, Equation 3, can be reduced to a set of equations which contain coupling only in the nonlinear terms. This requires the determination of the eigenvalues and eigenvectors of the undamped linear system

$$\omega_j^2[M]\{\phi^{(j)}\} = [K]\{\phi^{(j)}\} \qquad j = 1,2,\ldots,m \qquad (4)$$

in which ω_j is the natural frequency and $\{\phi^{(j)}\}$ is the corresponding j-th mode shape of the linear structure.

Apply a coordinate transformation, from the nodal displacements to the modal coordinates, by

$$\underset{m\times 1}{\{q\}} = \underset{m\times n}{[\phi]}\underset{n\times 1}{\{\xi\}} \qquad n \leq m \qquad (5)$$

in which $\{\xi\}$ represents a vector of modal coordinates. Substituting Equation 5 into Equation 3 and premultiplying by the transpose of $[\phi]$, Equation 3 becomes

$$\lceil M \rfloor\{\ddot{\xi}\} + [\phi]^T[C][\phi]\{\dot{\xi}\} + \lceil K \rfloor\{\xi\}$$
$$+ [\phi]^T[K^g(\{\xi\})][\phi]\{\xi\} = \{P(t)\} \qquad (6)$$

where $\{P\} = [\phi]^T \{F\}$ is the generalized force vector in modal coordinates. The terms M_j and K_j are the j-th generalized mass and stiffness defined by

$$M_j = \{\phi^{(j)}\}^T[M]\{\phi^{(j)}\}$$

$$K_j = \{\phi^{(j)}\}^T[K]\{\phi^{(j)}\} = \omega_j^2 M_j \qquad j = 1,2,\ldots,n \qquad (7)$$

The equations of motion, Equation 6, will contain coupling only in the nonlinear terms if the viscous damping (Reference 29) is proportional to inertia, stiffness, or both, that is

$$[C] = \mu[M] + \lambda[K] \qquad (8)$$

where μ and λ are proportionality constants. Then the j-th generalized damping coefficient is given by

$$C_j = \{\phi^{(j)}\}^T[C]\{\phi^{(j)}\}$$

$$= \mu M_j + \lambda K_j \qquad j = 1,2,\ldots,n \qquad (9)$$

Structural damping is another form of damping that allows it to be uncoupled, that is

$$[C] = ig[K] \qquad (10)$$

where $g(g \ll 1)$ is the structural damping coefficient. The j-th generalized damping coefficient is

$$C_j = ig\, K_j \qquad (11)$$

Sometimes it is more convenient to represent the damping as a fraction of critical damping. The modal damping ratio ζ_j represents the fraction of critical damping in the j-th mode. This ratio is

related to viscous damping proportionality constants μ and λ in Equation 8, and to structural damping coefficient g in Equation 10 by the relationships

$$2\zeta_j = \begin{cases} \dfrac{\mu}{\omega_j} + \lambda\omega_j, \text{ for viscous damping} \\ g, \text{ for structural damping} \end{cases} \tag{12}$$

When damping can be represented by proportional viscous or structural damping, then the equations of motion, Equation 6 can be written as

$$[M]\{\ddot{\xi}\} + [C]\{\dot{\xi}\} + [K]\{\xi\} + [\phi]^T[K^g(\{\xi\})][\phi]\{\xi\} = \{P\} \tag{13}$$

The j-th row of Equation 13 has the form

$$\begin{aligned} M_j\ddot{\xi}_j + C_j\dot{\xi}_j + K_j\xi_j + \sum_{k=1}^{n} \{\phi^{(j)}\}^T[K^g]\{\phi^{(k)}\}\xi_k &= P_j \\ M_j\ddot{\xi}_j + C_j\dot{\xi}_j + K_j\xi_j + \sum_{k=1}^{n} \xi_k \sum_{r=1}^{m} \sum_{s=1}^{m} \Phi_{rj}K^g_{rs}\Phi_{sk} &= P_j \end{aligned} \tag{14}$$

which has coupling only in the nonlinear term.

3. EQUIVALENT LINEARIZATION APPROACH

The basic idea of the equivalent linearization method (References 14 and 17) is to replace the actual system, Equation 13 or 14, with a set of equations of the form

$$M_j\ddot{\xi}_j + C_j\dot{\xi}_j + K_j^{eq}\xi_j + e_j(\xi_1,\xi_2,\ldots,\xi_n) = P_j \qquad j = 1,2,\ldots,n \tag{15}$$

where K_j^{eq} is an equivalent linear stiffness constant, and e_j is the error of linearization or equation deficiency term.

If this error term e_j is neglected, then Equation 15 is linear and it can be readily solved. The smaller that the error is, the smaller the error in neglecting it, and the better approximate solution

to Equation 14 will be obtained. To this end, the n equivalent linear stiffness constants, k_j^{eg}, are chosen in such a way that the mean-square, $E[e_j^2]$, is minimized. The error of linearization is

$$e_j = K_j\xi_j - K_j^{eq}\,\xi_j + \sum_{k=1}^{n} \{\phi^{(j)}\}^T[K^g(\{\xi\})]\{\phi^{(k)}\}\xi_k \qquad j = 1,2,\ldots,n \tag{16}$$

which is the difference between Equation 14 and Equation 15. From Equation 16 it is apparent that e_j depends upon the equivalent linear stiffness constant K_j^{eq}. It is these constants which will vary in order to minimize the n values of $E[e_j^2]$ requiring the following equation for K_j^{eq}

$$\frac{\partial E[e_j^2]}{\partial K_j^{eq}} = 0 \qquad j = 1,2,\ldots,n \tag{17}$$

where the operator E[ ] denotes the statistical average or mathematical expectation of the appropriate variables. Substituting Equation 16 into Equation 17 and interchanging the order of differentiation and expectation Equation 17 reduces to

$$K_jE[\xi_j^2] - K_j^{eq}E[\xi_j^2] + E[\xi_j \sum_{k=1}^{n} \{\phi^{(j)}\}^T[K^g(\{\xi\})]\{\phi^{(k)}\}\xi_k] = 0 \qquad j = 1,2,\ldots,n \tag{18}$$

Solving for the equivalent linear stiffness constant K_j^{eq}, Equation 18 gives

$$K_j^{eq} = K_j + \frac{E[\xi_j \sum_{k=1}^{n} \{\phi^{(j)}\}^T[K^g(\{\xi\})]\{\phi^{(k)}\}\xi_k]}{E[\xi_j^2]} \qquad j = 1,2,\ldots,n \tag{19}$$

Note that Equation 19 is not an explicit equation for K_j^{eq}, since the expectations appearing on the right-hand side depend on K_j^{eq}. In addition, the geometrical stiffness matrix which is needed in the numerator is not known a priori, and the coupling of the modal displacements causes some difficulties in evaluating the expectation. Therefore, one has to turn to simpler approach with a single-mode approximation solution. Further study is needed to evaluate Equation 19 numerically or by some other means.

4. SINGLE-MODE APPROACH AND MEAN-SQUARE RESPONSES

For most of the sonic fatigue analyses in practice, only a single-mode approach and linear small deflection theory are commonly employed. The inclusion of the large deflection into the analysis with a single-mode approximation represents a significant improvement of the design tools for complex structural panels.

Thus, if a single-mode approximation (usually the fundamental mode) is assumed, if the excitation pressure to the structure is stationary, Gaussian, and has a zero mean, then the expectations in Equation 19 can be evaluated, and the equivalent linear stiffness constant becomes

$$K^{eq} = K + 3\{\phi\}^T[K^g]\{\phi\}E[\xi^2] \tag{20}$$

in which the subscript j has been dropped. Note again that Equation 20 is not explicit for K^{eq}, since the expectation $E[\varepsilon^2]$ depends on K^{eq}, and also the geometrical stiffness matrix which is displacement dependent and is not known a priori. Therefore, an iterative scheme is introduced to determine K^{eq} from Equation 20. This will be presented later. At present, let us assume that a satisfactory equivalent linear stiffness constant has been found. By dropping the error of linearization $e(\xi)$ from Equation 15, the single-mode approximation solution of Equation 3 is obtained from

$$M\ddot{\xi} + C\dot{\xi} + K^{eq}\xi = P \tag{21}$$

or

$$\ddot{\xi} + 2\zeta\omega\dot{\xi} + \omega_{eq}^2 \xi = \frac{P}{M} \tag{22}$$

where $\omega_{eq} = (K^{eq}/M)^{1/2}$ is an equivalent linear frequency, and ζ is the modal damping ratio related to the linear frequency ω by Equation 12.

Now a random analysis of the modal equation (22) may be easily carried out to yield the PSD for the modal amplitude ξ as

$$S_\xi(\Omega) = S_p(\Omega)|H(\Omega)|^2 = \{\phi\}^T[S_F(\Omega)]\{\phi\}|H(\Omega)|^2 \tag{23}$$

in which $S_F\ (\Omega)$ is the cross spectral density matrix of the noise excitation $\{F\}$. The frequency response function $H(\Omega)$ is given by

$$H(\Omega) = \frac{1}{M(\omega_{eq}^2 - \Omega^2 + i2\zeta\omega\ \Omega)} \tag{24}$$

The mean-square response of modal amplitude is related to $S_\xi(\Omega)$ by

$$\begin{aligned} E[\xi^2] &= \int_0^\infty S_\xi(\Omega)\ d\Omega \\ &= \int_0^\infty \frac{S_p(\Omega)d\Omega}{M^2[(\omega_{eq}^2 - \Omega^2)^2 + (2\zeta\omega\ \Omega)^2]} \end{aligned} \tag{25}$$

For lightly damped ($\zeta \leq 0.05$) structures, the response curves will be highly peaked at ω_{eq}. The integration of Equation 25 can be greatly simplified if $S_p\ (\Omega)$ or $[S_F\ (\Omega)]$ can be considered to be constant in the frequency band surrounding the resonance peak, so that

$$\begin{aligned} E[\xi^2] &= S_p(\omega_{eq}) \int_0^\infty |H(\Omega)|^2 d\Omega \\ &= \frac{\pi\ S_p(\omega_{eq})}{4\ M^2\omega\ \omega_{eq}^2\ \zeta} \end{aligned} \tag{26}$$

The covariance matrix of the nodal deflections {q} can be obtained by use of the coordinate transformation given in Equation 15, then

$$[\overline{q_j q_k}] = \{\phi\}\ E[\xi^2]\{\phi\}^T \qquad j,k = 1,2,\ldots,m \tag{27}$$

The diagonal elements of $[q_j q_k]$ are the mean-square values of the nodal deflections, and the off-diagonal terms are time averages of products of deflections at different nodes. The mean-square node deflection is simply

$$\begin{aligned} \overline{q_j^2} &= E[\xi^2]\ \phi_j^2 \\ &= \frac{\pi\phi_j^2\ S_p(\omega_{eq})}{4M^2\omega\ \omega_{eq}^2\zeta} \qquad j = 1,2,\ldots m \end{aligned} \tag{28}$$

in which ϕ_j is the j-th element in modal vector $\{\phi\}$.

The element strains and nodal deflections are related by

$$\{\varepsilon\} = [S]_1\{q\} + [S]_2\{q\} \tag{29}$$

in which $\{\varepsilon\}$ is the vector of element strains. $[S]_1$ and $[S]_2$ are the strain-deflection transformation matrices. $[S]_1$ is the usual transformation matrix based on linear theory, and $[S]_2$ represents the inplane strains due to large deflections. Equation 29 is based on an appropriate linearization of the nonlinear strain-displacement relations (Reference 30) which will be discussed in Section V. Thus, the strain covariance matrix is given by

$$\begin{aligned} [\overline{\varepsilon_r\varepsilon_s}] &= [S]_1[\overline{q_j q_k}][S]_1^T + [S]_1[\overline{q_j q_k}][S]_2^T \\ &\quad + [S]_2[\overline{q_j q_k}][S]_1^T + [S]_2[\overline{q_j q_k}][S]_2^T \end{aligned} \tag{30}$$

The diagonal elements are the mean-square values of element strains.

5. ITERATION PROCEDURE AND FLOW CHART

As it was pointed out earlier, in Equation 20 the equivalent linear stiffness constant K^{eq} depends upon its response, which, in turn, depends upon K^{eq}. And also, the geometrical stiffness matrix $[k^g]$ is response dependent and it is not known a priori. In this way an iterative approach to final solution occurs. It makes no difference at which point in the iteration cycle the process begins. One certainly can assume either equivalent linear stiffness or response at the outset. The process will converge to an answer, provided the structure considered is a stable one. All panel structures of aircraft or space vehicle under sound environment are examples of such stable types.

Suppose, for definiteness, one desires to estimate the initial equivalent linear stiffness constant as $K_1{}^{eq}$. The fact that both $E[\xi^2]$ and $[K^g]$ can be approximated using the solutions of the linear equations of motion (in Equation 26, ω_{eq} has to be replaced by ω in Equation 4), facilitates the initial estimate of K^{eq} through Equation 20 as

$$K_1^{eq} = K + 3\{\phi\}^T[K^g]_o\{\phi\}E[\xi^2]_o \qquad (31)$$

This calculated initial estimate of $K_1{}^{eq}$ can be used to obtain refined estimate of $E[\xi^2]_1$ and $[K^g]_1$, that implies $K_2{}^{eq}$ through Equation 20 in the same way that Equation 31 implied $K_1{}^{eq}$. As the iterative process converges on the j-th cycle, the relation

$$\begin{aligned} K_j^{eq} &= K + 3\{\phi\}^T[K^g]_{j-1}\{\phi\}E[\xi^2]_{j-1} \\ &\cong K_{j-1}^{eq} \end{aligned} \qquad (32)$$

becomes satisfied. The number of cycles required to attain convergence depends on the nonlinear characteristics of the structure $[K^g]$, the intensity of the excitation $[S_p\ (\omega_{eq})]$, and the accuracy desired. The solution procedure is illustrated by a simplified flow chart shown in Figure 2.

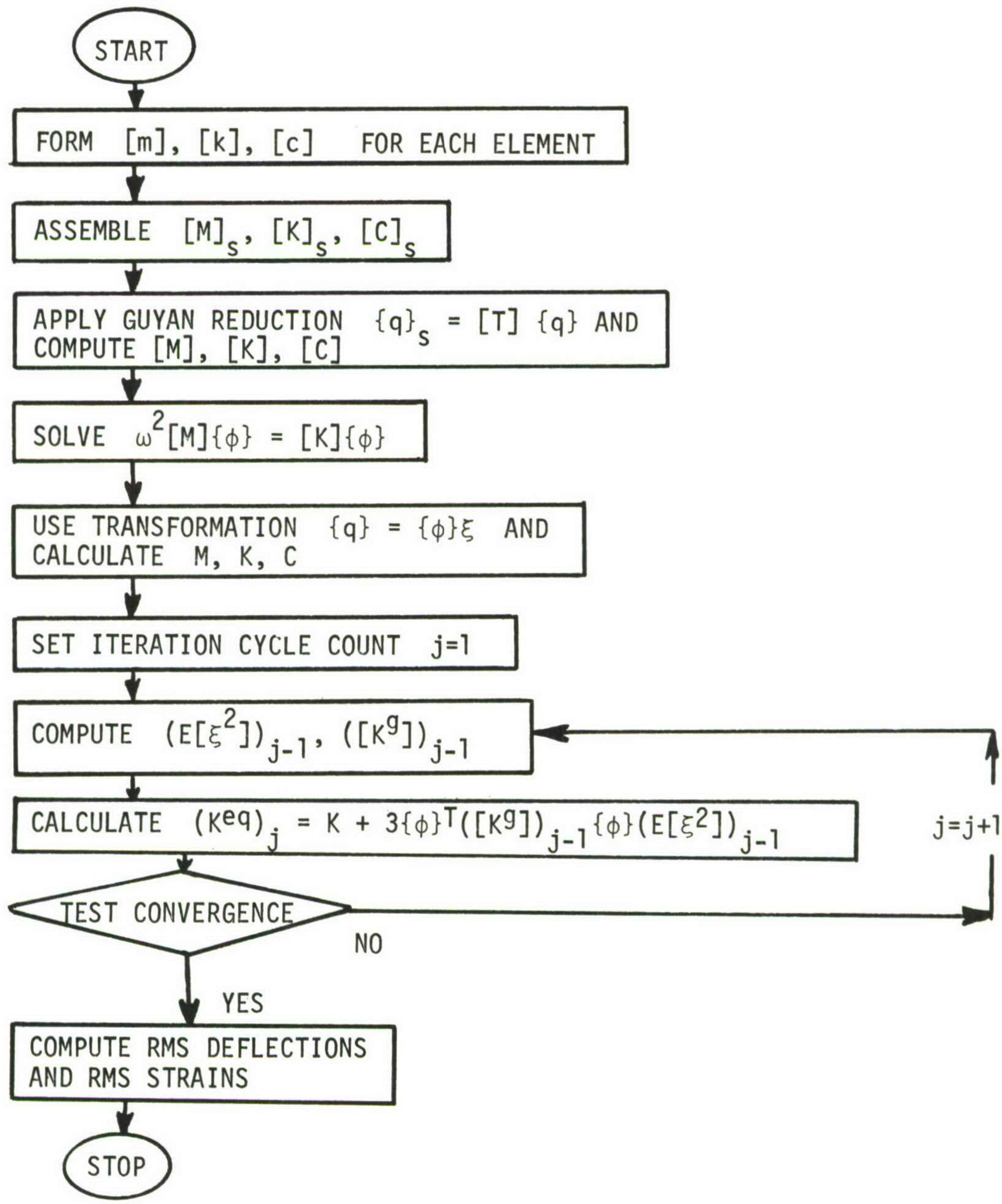


Figure 2. Simplified Flow Chart for Complex Panel Nonlinear Response to High Intensity Random Loads

6. ESTIMATION OF IMPROVEMENT

It is a very difficult task to give an accurate quantitative estimate on the improvement that would be made in predicting random responses using the nonlinear formulation without developing the computer program and analyzing these problems given in Section II. A preliminary estimate of root-mean-square (RMS) deflections, however, is possible. The ratio of RMS deflection based on large deflection nonlinear theory to RMS deflection using linear theory is given by

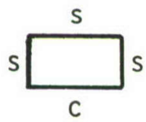
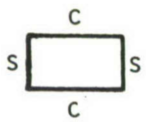
$$\frac{(\text{RMS deflection})_{NL}}{(\text{RMS deflection})_{L}} = \frac{\omega}{\omega_{eq}} \sqrt{\frac{S_p(\omega_{eq})}{S_p(\omega)}} \tag{33}$$

in which ω is the linear natural frequency, and ω_{eq} is the equivalent linear frequency. If the spectral density function of excitation $S_p(\Omega)$ is a slowly varying function with respect to frequency, Equation 33 can be further simplified to

$$\frac{(\text{RMS deflection})_{NL}}{(\text{RMS deflection})_{L}} \cong \frac{\omega}{\omega_{eq}} \tag{34}$$

Let us examine the data obtained in Reference 6 (see Section II, paragraph 2), because this was the only report in which deflections were measured. For a Skin-Stringer Panel, the measured RMS deflection is 0.064 inch, or twice the skin thickness. Comparison showed that calculated panel deflection based on linear theory is 34-percent high. The rectangular panel has a length-to-width ratio of 3.7. It was assumed to be clamped along the edges in the analysis. The ratio

of linear frequency to equivalent linear frequency needed in Equation 33 or Equation 34 can be estimated from the following data:

Support Condition	Length-to-Width Ratio	Reference	$\frac{\omega}{\omega_{eq}}$
Clamped	1.0	31	0.64
Clamped	2.0	30	0.58 (extrapolated)
	3.7	32	0.53
	3.7	32	0.65

With an approximate ratio of $(\omega/\omega_{eq}) \cong 0.6$, Equation 33 becomes

$$\frac{(\text{RMS deflection})_{NL}}{(\text{RMS deflection})_{L}} \cong 0.6 \sqrt{\frac{S_p(\omega_{eq})}{S_p(\omega)}} \qquad (35)$$

The RMS deflection based on large deflection theory would be very close to the measured value.

Without analyzing the problem with a computer program, it is very difficult to give a quantitative estimate on the improvement in predicting response strains using nonlinear theory. The difficulties come from the transformation matrices $[S]_1$ and $[S]_2$ in Equation 30. Therefore, no attempt is given for pursuing it further.

SECTION V

GEOMETRICAL STIFFNESS MATRICES

The geometrical stiffness matrix $k^g(\{\delta\})$ needed in the formulation has been developed for various types of elements. They have been used successfully, in conjunction with the consistent mass and stiffness matrices, in problems of large amplitude vibrations of complex structures. A brief review of the advances in the development of geometrical stiffness matrices and the associated large amplitude vibration problems is given in this section.

Extension of the finite element method to large amplitude vibrations of beams and rectangular plates was first reported by Mei (References 33 and 34). The geometrical stiffness matrix formulation for a rectangular plate element in Reference 34 was based on a modified form of the Berger's hypothesis (Reference 35). Nonlinear frequencies, which were obtained for rectangular plates with various edge support conditions, agreed well with the approximate analytical solutions. Results for some boundary conditions in Reference 34 were obtained for the first time.

One important thing which has to be mentioned at this point is that the nonlinear frequencies determined from the large amplitude vibrational analysis using a "Quasi-Linearization" technique have the very same physical meaning, comparing Equation 14 with Equations 21 and 22, as the equivalent linear frequency, ω_{eq}, using equivalent linearization method. Therefore, gaining a better understanding of large amplitude vibrations of plate and shell structures will certainly be helpful in building a more solid background for problems of random vibrations of complex nonlinear structures. This has been found true in the linear case.

Recently, Rao and his colleagues presented a simplified formulation for large amplitude vibrations of beams (Reference 36), rectangular plates (References 30 and 37), and circular plates (Reference 38). Their formulation is based on an appropriate linearization of the nonlinear strain-displacement relations and an iterative scheme of Mei's (Reference 33) to obtain the nonlinear frequencies. This linearization

technique was used in the strain-deflection relations of Equation 29. Reddy and Stricklin (Reference 39) developed a linear and a quadratic isoparametric rectangular element to study large amplitude plate vibrations. Most recently, two triangular element formulations have been developed for large amplitude vibrations of thin plates of arbitrary planform. The first one (Reference 40) is consistent with the higher order bending element TRPLTI (Reference 41) in NASTRAN program, and the second (Reference 41) is consistent with the high precision plate element of Cowper et al. (Reference 43). Nonlinear frequencies obtained for numerical examples include rectangular, circular, rhombic, and isosceles triangular plates.

Raju and Rao (References 44 and 45) intended to develop a shell of revolution frustum for nonlinear vibration analysis of thin shells of revolution; however, their results failed to predict the "softening" type of nonlinear behavior as discussed by Evensen (Reference 46).

Geometrical stiffness matrices of various types of finite elements that have been developed for free vibration analysis involving large deflection nonlinearities are listed in Table 5.

TABLE 5

GEOMETRICAL STIFFNESS MATRICES
OF VARIOUS FINITE ELEMENTS

Element Type	Reference
Beam	Mei[33] (1972)
	Rao et al.[36] (1976)
Rectangular Plate	Mei[34] (1973)
	Rao et al.[30] (1976)
	Reddy and Stricklin[39] (1977)
Rectangular (Orthotropic) Plate	Rao et al.[37] (1976)
Circular Ring (Orthotropic) Plate	Rao et al.[38] (1976)
Triangular Plate	Mei and Rogers[40] (1977)
	Mei et al.[42] (1978)
Shell of Revolution	Raju and Rao[44,45] (1975, -76)

SECTION VI

QUASI-LINEARIZATION METHOD

The same physical interpretation between the equivalent linear frequency and the nonlinear frequency, Equations 14, 21, and 22, leads to the idea that the method of quasi-linearization may also be applied to problems of complex nonlinear structures subjected to random loads. The quasi-linearization technique has been used successfully in predicting nonlinear frequencies for complex structures (References 33, 34, 36-40, 42, 44, 45). Nonlinear frequencies of higher modes can also be determined by this technique (References 30, 33, 40 and 42). Those nonlinear frequencies, ω_{eq}, j, can be employed to obtain an approximate random response of deflections from the relation

$$[\overline{q_r q_s}] = \sum_{j=1}^{n} \{\phi^{(j)}\} \frac{\pi\{\phi^{(j)}\}^T [S_F(\omega_{eq,j})]\{\phi^{(j)}\}}{4M_j^2 \omega_j \omega_{eq,j}^2 \zeta_j} \{\phi^{(j)}\}^T \qquad (36)$$

$$r,s=1,2,\ldots,m$$

This method may very well be more promising than the equivalent linearization technique. However, both methods are worth pursuing further.

SECTION VII

CONCLUSIONS

1. SUMMARY OF RESULTS

The omission of "large amplitude nonlinearity" in the analysis is identified as the major factor that contributed to the wide discrepancy between the measured test data and computed results. A mathematical formulation based on the finite element displacement method and the equivalent linearization technique is developed. Statistical responses using a single-mode approximation can be expressed in terms of linear frequency, spectral density of excitation, equivalent linear frequency, and transformation matrices. An iterative scheme for determining the equivalent linear stiffness constant is presented. Advances in the development of geometrical stiffness matrices for various finite elements are reviewed. Finally, a concept of applying the quasi-linearization method to problems of nonlinear complex panel structures subjected to high intensity noise levels is discussed briefly.

2. RECOMMENDATIONS

a. Development of Computer Program. A computer program should be developed based on the finite element-equivalent linearization formulation presented. The computed results should be compared with the experiments.

b. Experiments. Carefully monitored and controlled experiments with simple structures (such as plates) and typical aircraft panels should be conducted at both low and high noise environment. Measurements of deflection, strain, frequency, and pressure spectral density should be precisely recorded.

c. Refinement of Finite Elements. Refined finite element representations should be developed and incorporated into the computer program; such refinements, for example, as higher-order displacement

function for beam element with various thin-walled cross-sections, variation in thickness for plate element, shallow shell element, anisotropic properties for modeling composite panels, etc.

d. Quasi-Linearization Method. More research study on this very promising approach should be conducted.

REFERENCES

1. Thomson, A. G. R. and Lambert, R. F., "Acoustic Fatigue Design Data", AGARD-AG-162-Part I and II, NATO Advisory Group for Aero. Res. and Dev., 1972.

2. Rudder, F. F., and Plumblee, H. E., Jr., "Sonic Fatigue Design Guide for Military Aircraft", AFFDL-TR-74-112, W-PAFB, OH, 1975.

3. Ungar, E. E. et al., "A Guide for Estimation of Aeroacoustic Loads on Flight Vehicle Surfaces", AFFDL-TR-76-91, W-PAFB, OH, 1977.

4. Wentz, K. R. and Wolfe, H. F., "Development of Random Fatigue Data for Adhesively Bonded and Weldbonded Structures Subjected to Dynamic Excitation", ASME Journal of Engineering Materials and Technology, Vol. 100, Jan 1978, pp. 70-76.

5. Fitch, G. E. et al., "Establishment of the Approach to, and Development of Interim Design Criteria for Sonic Fatigue", ASD-TDR-62-26, W-PAFB, OH 1962, pp. 42-46.

6. Jacobs, L. D. and Lagerquist, D. R., "Finite Element Analysis of Complex Panel to Random Loads", AFFDL-TR-68-44, W-PAFB, OH, 1968, pp. 55-59.

7. Jacobson, M. J., "Advanced Composite Joints; Design and Acoustic Fatigue Characteristics", AFFDL-TR-71-126, W-PAFB, OH, 1972, pp. 133 and 208.

8. Jacobson, M. J., "Sonic Fatigue Design Data for Bonded Aluminum Aircraft Structures", AFFDL-TR-77-45, W-PAFB, OH, 1977, pp-54-57.

9. Caughey, T. K., "Derivation and Application of the Fokker-Planck Equation to Discrete Nonlinear Dynamic Systems Subjected to White Random Excitation", JASA, Vol. 35, Nov. 1973, pp. 1683-1692.

10. Caughey, T. K., "Nonlinear Theory of Random Vibrations", Advances in Applied Mechanics, Edited by Yih, C. S., Vol. 11, Academic Press, 1971, pp. 209-253.

11. Herbert, R. E., "Random Vibrations of a Nonlinear Elastic Beam", JASA, Vol. 36, Nov. 1964, pp. 2090-2094.

12. Herbert, R. E., "Random Vibrations of Plates with Large Amplitude", J. Applied Mechanics, Vol. 32, Sept. 1965, pp. 547-552.

13. Krylov, N. and Bogoliubov, N., "Introduction to Nonlinear Mechanics", translated by Lefshetz, S. in Annals of Mathematical Studies, No. 11, Princeton University Press, N. J., 1943.

14. Caughey, T. K., "Equivalent Linearization Techniques", JASA, Vol. 35, 1963, pp. 1706-1711.

REFERENCES (Cont'd)

15. Lin, Y. K., "Response of a Nonlinear Flat Panel to Periodic and Randomly-Varying Loadings", J. Aerospace Sci., Sept. 1962, pp. 1029-1033, p. 1066.

16. Seide, P., "Nonlinear Stresses and Deflections of Beams Subjected to Random Time Dependent Uniform Pressure", ASME Paper No. 75-DET-23.

17. Foster, E. T., Jr., "Semilinear Random Vibrations in Discrete Systems", J. Applied Mechanics, Vol. 35, Sept. 1968, pp. 560-564.

18. Iwan, W. D. and Yang, I. M., "Application of Statistical Linearization Technique to Nonlinear Multidegree-of-Freedom Systems", J. Applied Mechanics, Vol. 39, June 1972, pp. 545-550.

19. Foster, E. T., Jr., "Predicting Wave Responses of Deep-Ocean Towers", Proceedings of the Conference on Civil Engineering in the Oceans, ASCE, 1968, pp. 75-98.

20. Crandall, S. H., "Perturbation Techniques for Random Vibration of Nonlinear Systems", JASA, Vol. 35, Nov. 1963, pp. 1700-1705.

21. Lyon, R. H., "Response of a Nonlinear String to Random Excitation", JASA, Vol. 32, August 1960, pp. 953-960.

22. Tung, C. C., Penzien, J. and Horonjeff, R., "The Effect of Runway Uneveness on the Dynamic Response of Supersonic Jet Transport", NASA CR-119, University of California, Berkeley, 1964.

23. Fox, H. L., Smith, P. W., Jr., Pyle, R. W. and Nayak, P. R., "Contributions to the Theory of Randomly Forced, Nonlinear, Multiple-Degree-of-Freedom, Coupled Mechanical Systems", AFFDL-TR-72-45. W-PAFB, OH, 1973.

24. Belz, D. L., "A Numerical Study of Nonlinear Vibrations Induced By Correlated Random Loads", Int. J. Nonlinear Mechanics, Vol. 1, 1966, pp. 139-145.

25. Jacobs, L. D. and Lagerquist, D. R., "A Finite Element Analysis of Simple Panel Response to Turbulent Boundary Layers", AFFDL-TR-67-81, W-PAFB, OH, 1967.

26. Olson, M. D. and Lindberg, G. M., "Free Vibrations and Random Response of an Integrally-Stiffened Panel", National Research Council of Canada Report LR-544, Ottawa, Canada, 1970.

27. Olson, M. D., "A Consistent Finite Element Method for Random Response Problems", Computers and Structures, Vol. 2, 1972, pp. 163-180.

REFERENCES (Cont'd)

28. Penzien, J., Kaul, M. K., and Berge, B., "Stochastic Response of Off-Shore Towers to Random Sea Waves and Strong Motion Earthquakes", Computers and Structures, Vol. 2, 1972, pp. 733-756.

29. Hurty, W. C. and Rubinstein, M. F., "Dynamics of Structures", Prentice-Hall, 1964.

30. Rao, G. V., Raju, I. S., and Raju, K. K., "A Finite Element Formulation for Large Amplitude Flexural Vibrations of Thin Rectangular Plates", Computers and Structures, Vol. 6, 1976, pp. 163-167.

31. Yamaki, N., "Influence of Large Amplitudes on Flexural Vibrations of Elastic Plates", ZAMM, Vol. 41, 1961, pp. 501-510.

32. Wah, T., "Large Amplitude Flexural Vibration of Rectangular Plates", Int. J. Mech. Sci., Vol. 5, 1963, pp. 425-438.

33. Mei, C., "Nonlinear Vibration of Beams by Matrix Displacement Method", AIAA J., Vol. 10, March 1972, pp. 355-357.

34. Mei, C., "Finite Element Displacement Method for Large Amplitude Free Flexural Vibrations of Beams and Plates", Computers and Structures, Vol. 3, 1973, pp. 163-174.

35. Berger, H. M., "A New Approach to the Analysis of Large Deflections of Plates", J. Applied Mechanics, Vol. 22, December 1955, pp. 465-472.

36. Rao, G. V., Raju, I. S. and Raju, K. K., "Nonlinear Vibrations of Beams Considering Shear Deformation and Rotary Inertia", AIAA J., Vol. 14, May 1976, pp. 685-687.

37. Raju, K. K. and Rao, G. V., "Nonlinear Vibrations of Orthotropic Plates by a Finite Element Method", J. Sound Vib., Vol. 48, 1976, pp. 301-303.

38. Rao, G. V., Raju, I. S. and Raju, K. K., "Finite Element Formulation for the Large Amplitude Free Vibrations of Beams and Orthotropic Circular Plates", Computers and Structures, Vol. 6, 1976, pp. 169-172.

39. Reddy, J. N. and Stricklin, J. D., "Large Deflection and Large Amplitude Free Vibrations of Thin Rectangular Plates using Mixed Isoparametric Elements", Symp. on Applications of Computer Methods in Engineering, University of Calif., Los Angeles, Calif, August 23-26, 1977.

REFERENCES (Cont'd)

40. Mei, C. and Rogers, J. R., Jr., "Application of the TRPLTI Element to Large Amplitude Free Vibrations of Plates", NASA CP-2018, Oct. 1977, pp. 275-298.

41. Narayanaswami, R. and Mei, C., "Addition of Higher Order Plate Elements to NASTRAN", NASA TM X-3428, Oct. 1976, pp. 439-477.

42. Mei, C., Narayanaswami, R., and Rao, G. V., "Large Amplitude Free Flexural Vibrations of Thin Plates of Arbitrary Shape", Computers and Structures (to appear).

43. Cowper, G. R., Kosko, E., Lindberg, G. M. and Olson, M. D., "Static and Dynamic Applications of a High-Precision Triangular Bending Element", AIAA J., Vol. 7, 1969, pp. 1957-1965.

44. Raju, K. K. and Rao, G. V., "Calculation of Nonlinear Axisymmetric Vibrations of Thin Shells of Revolution by a Finite Element Method", J. Sound Vib., Vol. 38, 1975, pp. 505-509.

45. Raju, K. K. and Rao, G. V., "Large Amplitude Asymmetric Vibrations of Some Thin Shells of Revolution", J. Sound Vib., Vol. 44, 1976, pp. 327-333.

46. Evensen, D. A., "Comment on Large Amplitude Asymmetric Vibrations of Some Thin Shells of Revolution", J. Sound Vib., Vol. 52, 1977, pp. 453-455.

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